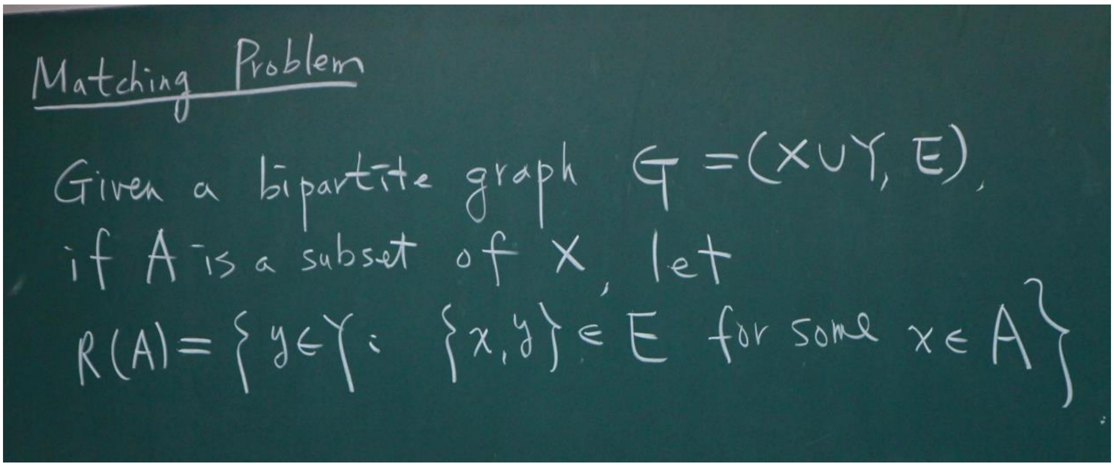
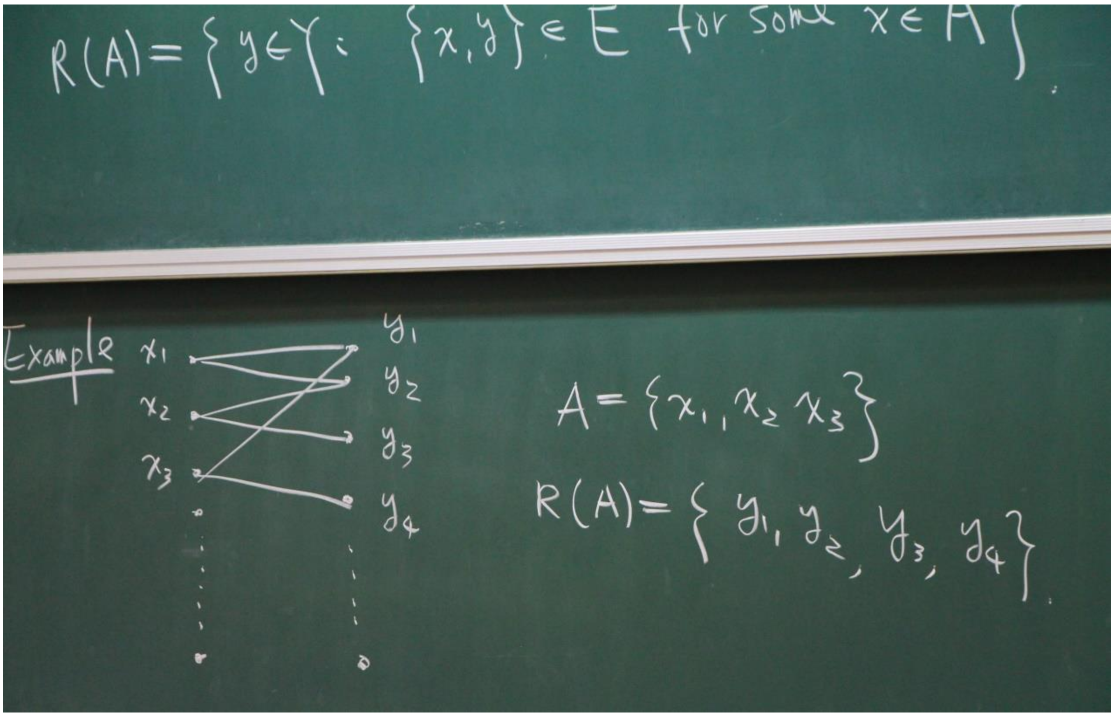
【10920 趙啟超教授離散數學 / 第 30 堂版書】

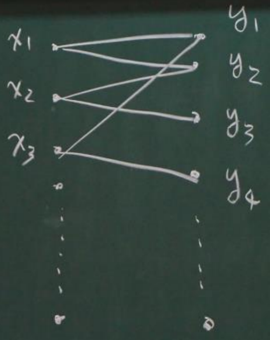




Matching Problem

Given a bipartite graph $G=(X\cup Y, E)$,
if A is a subset of X, let
$$R(A)=\{y\in Y : \{x,y\}\in E \text{ for some } x\in A\}.$$

Example



$$A=\{x_1, x_2, x_3\}$$
$$R(A)=\{y_1, y_2, y_3, y_4\}.$$

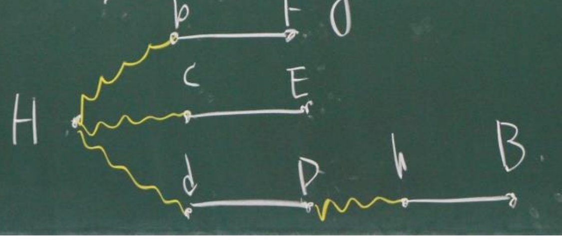
Theorem (Hall's Theorem)

The bipartite graph $G=(X\cup Y, E)$ has a complete matching iff $|R(A)|\geq |A|$ for all $A\subseteq X$.

Proof "$\Longrightarrow$" Trivial.

"$\Longleftarrow$" If a complete matching is impossible, the Hugarian algorithm will eventually produce a Hungarian tree, which contains one more X-vertex than Y-vertex, violating the condition.

E.g.,



<u>Theorem</u> (Hall's Theorem)

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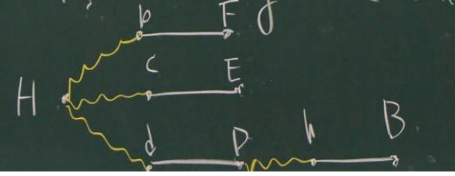

Def The deficiency δ of a bipartite graph $G = (X \cup Y, E)$ is defined as

$$\delta = \max_{A \subseteq X} \{ |A| - |R(A)| \}.$$

Remark If $A = \phi$, then $|A| - |R(A)| = 0 - 0 = 0$.

Consequently, $\delta \geq 0$.

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Property The size of a maximal matching M in a bipartite graph $G = (X \cup Y, E)$ is given by

$$|M| = |X| - \delta$$

where δ is the deficiency of G.

Proof By the definition of δ, there is a subset $A_0 \subseteq X$ for which $|A_0| - |R(A_0)| = \delta$.

In any matching M, at least δ members (of A_0) remain unmatched, and $|M| \leq |X| - \delta$.

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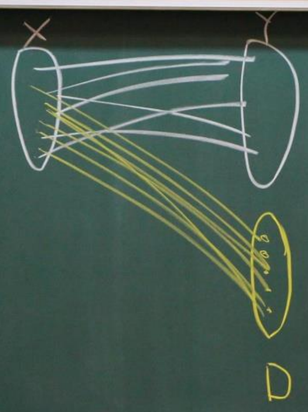

Let D be a new set of vertices of size δ, and let G^* be the graph $(X^* \cup Y^*, E^*)$ given by $X^* = X$, $Y^* = Y \cup D$, $E^* = E \cup K$ where K is the set of all possible edges, connecting X and D.

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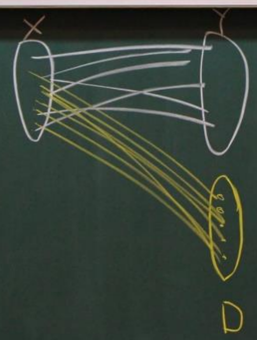

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Hence for all $A \subseteq X$ $(= X^*)$,

$$R^*(A) = R(A) \cup D \qquad \delta = \max_{A \subseteq X} \{|A| - |R(A)|\}$$

and thus

$$|R^*(A)| - |A| = |R(A)| + |D| - |A|$$
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